Journal of Engineering Physics and Thermophysics, Vol. 68, No. 1, 1995

CALCULATION OF THE TEMPERATURE OF A HEAT-TRANSMITTING SURFACE FROM THE RESULTS OF INDIRECT MEASUREMENTS

A. S. Trofimov, V. V. Kryzhnii, and
E. P. Kryzhnyaya

UDC 536.26

The problem of the identification of the non-stationary temperature regime of a heat-transmitting wall surface from the results of measurements at an inner point of the body or at a heat-insulated surface is considered. The basis for this method is the joint application of Lavrent'ev regularization and the method of imaginary frequency responses. The given approach enables one to obtain an expression in a form convenient for calculations.

In actual practice, problems of identification of the evolution law of an air stream or the temperature of a heat-transferring surface that is inaccessible for direct measurements for a number of reasons frequently arise. Among such problems are, in particular, problems of monitoring thermoelastic strains in construction elements and thermal technological parameters. Such problems are known to be ill-posed since they require reconstruction of a signal (cause) from the results of measurements of the response (consequences). Physical ill-posedness of the problem results in its mathematical ill-posedness.

The existing methods for solving ill-posed inverse problems [1–3] possessing a number of definite advantages can be applied, however, mainly to research problems and imply application of powerful computers.

We will show that restricting the consideration to bodies of simple geometry allows one to simplify substantially the algorithm for solving inverse boundary thermoconductivity problems (ITP) that consist in identification of the boundary condition at one of the surfaces. In this case thermophysical characteristics of the material are considered to be known, whereas the initial temperature is considered to be constant. To construct the algorithm we will proceed from the relationship between the known $\varphi(\mathrm{Fo})$ and sought $g(\mathrm{Fo})$ functions given as the convolution equation

$$\varphi(\mathrm{Fo}) = \int_0^{\mathrm{Fo}} \Pi(\mathrm{Fo} - \tau)\, g(\tau)\, d\tau . \tag{1}$$

The relationship (1) is obtained when solving the corresponding direct thermoconductivity problem, for which function $g(\mathrm{Fo})$, which determines the boundary conditions, is considered to be known. The kernel of the equation (1) is the inverse transform of the transmission function that relates the Laplace transforms of the functions $\varphi(\mathrm{Fo})$ and $g(\mathrm{Fo})$ and is known from the solution of the direct problem [4].

In solving the inverse problem, equation (1) is a Volterra integral equation of the first kind with respect to the sought function $g(\mathrm{Fo})$. To solve this equation we will apply the Lavrent'yev regularization method [5, 6], which consists in replacing equation (1) with a Volterra equation of the second kind that is close to it:

$$\varphi(\mathrm{Fo}) = \int_0^{\mathrm{Fo}} \Pi(t - \tau)\, g(\tau)\, d\tau + \alpha g(\mathrm{Fo}) , \tag{2}$$

where α is the regularization parameter.

Kuban' State Technological University, Krasnodar, Russia. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 68, No. 1, pp. 156-159 January-February, 1995. Original article submitted December 1, 1992, revision submitted May 18, 1994.

Taking the Laplace transform of equation (2), we find the image of the sought function

$$g(p) = \varphi(p)/(\Pi(p) + \alpha). \tag{3}$$

The immediate conversion to the inverse transforms in equation (3) is a painstaking job due to the absence of a tabulated inverse transform. To find an approximate inverse transform we make use of the fact that the Lavrent'yev regularization method can be applied also in the case of an inexactly given kernel of the integral equation (1). To form the approximate kernel we use the method of imaginary frequency responses [7], which consists in substitution of a fractional rational function for the Laplace transform. Let the approximation $\Pi(p)$ of the transmission function be

$$\tilde{\Pi}(p) = \frac{b}{b + ap + p^2}, \tag{4}$$

where a and b are certain coefficients. Substituting the approximation (4) into the equation (3) and going back to the inverse transforms we obtain the working formula

$$g(\mathrm{Fo}) = \frac{1}{\alpha}\left[\varphi(\mathrm{Fo}) - \frac{b}{\alpha\omega}\int_0^{\mathrm{Fo}} e^{-\beta(\mathrm{Fo}-\tau)} \sin[\omega(\mathrm{Fo}-\tau)]\varphi(\tau)\,d\tau\right], \tag{5}$$

where $-\beta = a/2$; $\omega = \sqrt{\left(\frac{1+\alpha}{\alpha}b - \beta^2\right)}$.

To carry out calculations according to expression (5) one should find the optimum value of the regularization parameter α, which depends on the error in the experimental data and the accuracy of the representation of the kernel. Since in practice the experimental errors are frequently unknown, the following criterion was used to determine the regularization parameter [8]:

$$\| y_{\alpha_{i+1}} - y_{\alpha_i} \| = \min, \tag{6}$$

where $\alpha_{i+1} = \theta\alpha_i$, $0 < \theta < 1$.

The choice of such a criterion has a theoretical basis only for several classes of inverse problems; however its practical application appeared to be most suitable in the method under consideration since such a choice of the parameter allows one to find its value from results of processing of data from a single experiment. In the subsequent calculations α may be considered to be equal to the value found.

Consider an analytical solution of the problem in the case where the experimentally determined temperature at the surface of a thermo-insulated plate is the linear function of time $u(0, \mathrm{Fo}) = A\mathrm{Fo}$. In this case the transmission function is expressed as $\Pi(p) = 1/\mathrm{ch}\sqrt{p}$, whereas its approximation (4) takes the form

$$\tilde{\Pi}(p) = \frac{24}{24 + 12p + p^2}.$$

Calculating the integral I in equation (5) and taking into account the numerical value $\beta = 6$, we obtain for large Fourier numbers

$$I \approx \left[A\,\mathrm{Fo} - \frac{2A\beta}{\beta^2 + \omega^2}\right]\frac{\omega}{\beta^2 + \omega^2}.$$

Substituting the value found into equation (5) and taking into account the dependence $\omega = \omega(\alpha)$, we eventually find

$$u(1, \mathrm{Fo}) = \frac{A\,\mathrm{Fo}}{1 + \alpha} + \frac{0.5A}{(1 + \alpha)^2}. \tag{7}$$

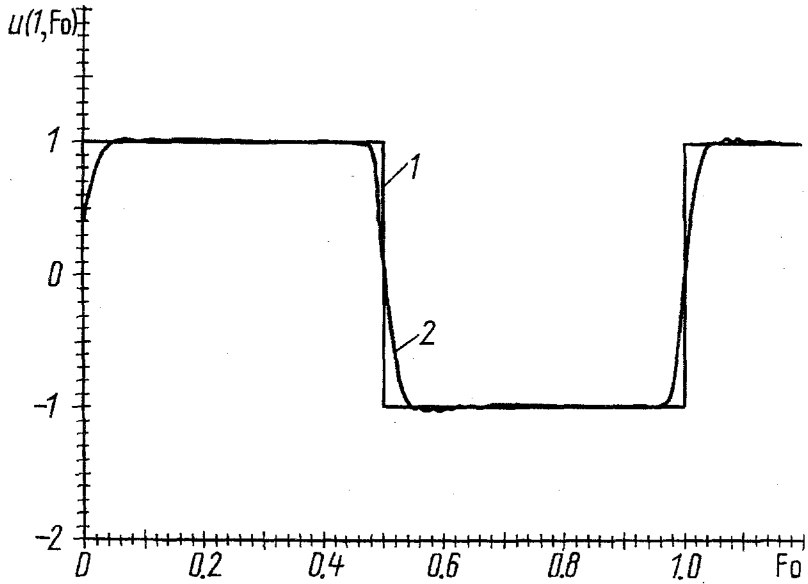

Fig. 1. Stepwise dependence of the boundary temperature on the Fo number (1) and its reconstruction (2).

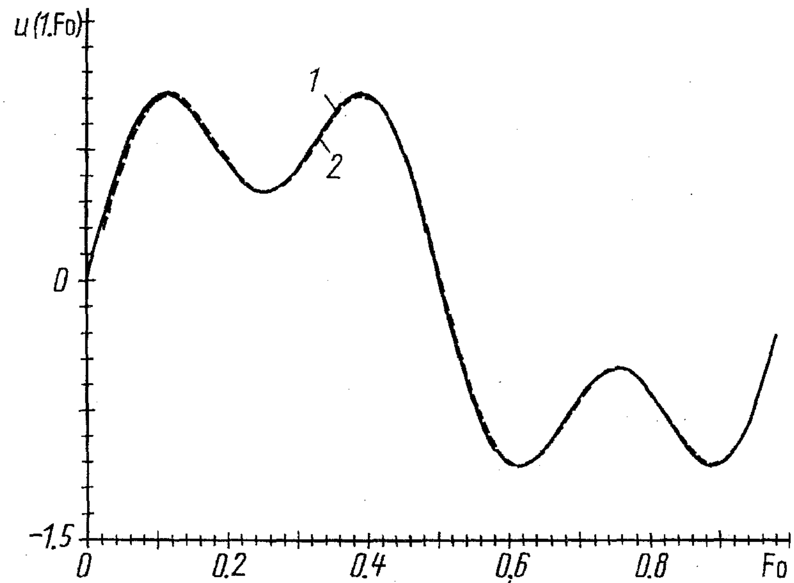

Fig. 2. Original dependence of the boundary temperature on the Fo number (1) and its reconstruction (2).

On the other hand, from an exact solution of the direct thermoconductivity problem with a linear evolution law of the boundary temperature, for large Fourier numbers [4] we obtain

$$u(1, \mathrm{Fo}) = u(0, \mathrm{Fo}) + 0.5A. \tag{8}$$

Comparing the dependences (7) and (8) one can easily see that at small values of the regularization parameter α they practically coincide.

We now dwell on the distinctive features of solving an inverse problem at small Fourier numbers when the signal delay time which arises from the error of measurements should be taken into account. Such a connection can be explained in the following manner. At time zero let the relative temperature at the outer surface of the plate be measured in the region from zero initial temperature to unity. If in this case the sensitivity of the detector allows one to measure the relative temperature vibrations at the termo-insulated surface with an accuracy of four decimal places, then, as follows from the results of solving the direct problem [4, p. 95], the signal will be recorded at a

Fourier number equal to 0.032, in the case of three decimal places this occurs at 0.04, in the case of two decimal places this occurs at 0.056, etc.

To test the possibilities of the method at low Fourier numbers a numerical experiment on the reconstruction of the stepwise dependence of the boundary temperature was carried out. The original data for the calculation were obtained by solving numerically the direct problem with a step along the Fourier number equal to 0.01 and rounding the results to the fourth decimal place. Figure 1 shows the original stepwise dependence and its reconstruction according to expression (5) with the reconstructed dependence plotted with a shift along the Fourier number equal to 0.032 in accordance with the accuracy of the original data.

To test the independence of the delay times from the shape of the original boundary dependence as well as to test the possibility of choosing the regularization parameter in the preliminary experiment the problem of reconstruction of the boundary temperature was solved:

$$g(\mathrm{Fo}) = \sin(2\pi\,\mathrm{Fo}) + 0.5\sin(6\pi\,\mathrm{Fo}).$$

The original data, as in the preceding example, were obtained by solving numerically the direct problem with a step along the Fourier number equal to 0.01 and rounding of the results to the fourth decimal place. Figure 2 shows the plots of the original and the reconstructed dependences with the latter given with a shift of 0.032. The regularization parameter α was taken to be equal in the calculations to the value found in the process of the reconstruction of the stepwise dependence.

The given method is rather general and can be applied to the reconstruction of boundary conditions of the first, second, and third kind for bodies of classical geometry. Such a generalization consists in the use of a transmission function in equation (3) and subsequent evaluation of its fractional rational approximation. In this case only the values of the coefficients are changed in the calculation formula (5). The practice of solving inverse problems has shown that the method considered in the present article is applicable in the case where the solution of the corresponding direct problem can be found with reasonable accuracy with the use of an approximation of the transmission function (4).

REFERENCES

1. O. M. Alifanov, Inverse Problems of Heat Transfer [in Russian], Moscow (1988).
2. O. M. Alifanov, E. A. Artyukhin, and S. V. Rumyantsev, Extremum Methods of Solving Ill-Posed Problems [in Russian], Moscow (1988).
3. A. N. Tikhonov, A. V. Goncharskii, V. V. Stepanov, and A. G. Yagola, Numerical Methods for the Solution of Ill-Posed Problems [in Russian], Moscow (1990).
4. A. V. Luikov, Theory of Thermoconductivity [in Russian], Moscow (1967).
5. M. M. Lavrent'yev, Dokl. Akad. Nauk SSSR, **127**, No. 1, 31-33 (1959).
6. A. F. Verlan' and V. S. Sizikov, Integral Equations. Methods, Algorithms, Programs [in Russian], Kiev (1986).
7. A. S. Trofimov, A. V. Kozlov, and E. Yu. Kovalenko, Inzh.-Fiz. Zh., **49**, No. 3, 513 (1985).
8. V. B. Glasko and G. V. Gushchin, Vychislitel'nye Metody i Programmirovanie, Issue 5, 61-73 (1966).